

Newton 중력 < 발견 - 절대진리 (x)
발명 - 사회적 합의 (o)

Einstein 시공간의 휘어짐

Physics

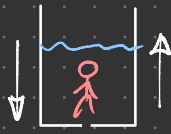
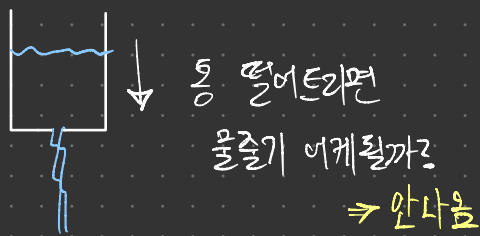
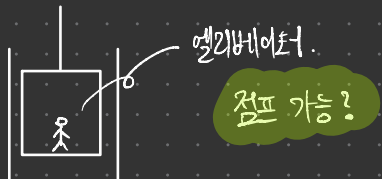
프린키피아

원제: 자연철학의 수학적 원리

why?

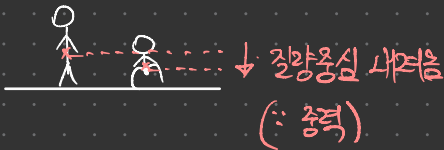
how?

나의 고민을 내가
해결할 수 있다.

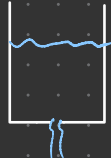


사람 영향: 무중력

무중력 상태에서 점프 가능?



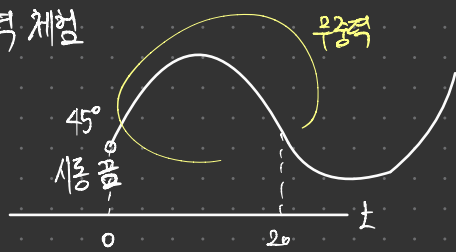
질량중심 위치 일정
=> 점프 불가능

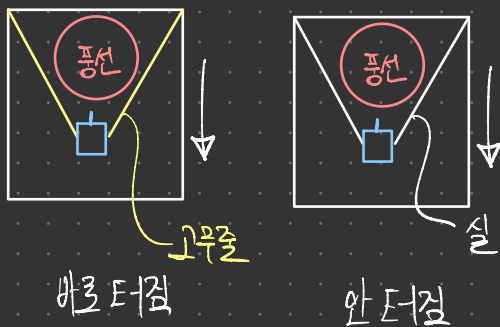


포물선 운동

=> 안나옴

무중력 체험





무중력 생각!

Chapter 09

Center of mass and linear Momentum

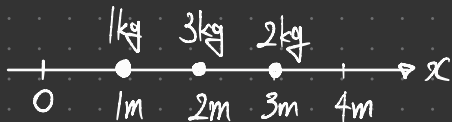
1. 질량중심

① 정의: 입자 위치의 질량에 대한 가중평균

Ex)

164cm 173cm 173cm 182cm

$$\text{평균 키} = \frac{164\text{cm} \times 1\text{명} + 173\text{cm} \times 2\text{명} + 182\text{cm} \times 1\text{명}}{4\text{명}}$$

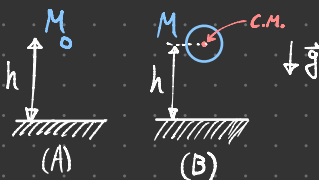


$$\text{평균 위치} = \frac{1\text{m} \times 1\text{kg} + 2\text{m} \times 3\text{kg} + 3\text{m} \times 2\text{kg}}{6\text{kg}}$$

$$\vec{r}_{cm} = \frac{\sum m_i \vec{r}_i}{\sum m_i}$$

예제) 중력퍼텐셜 에너지

(gravitational potential energy)



$$U_A = Mgh$$

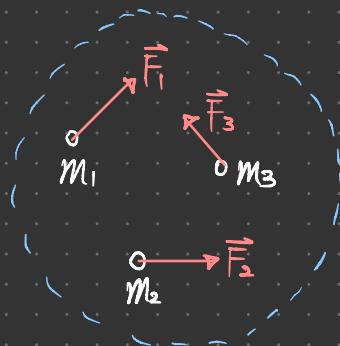
$$U_B = m_1gh_1 + m_2gh_2 + \dots$$

$$= \frac{\sum m_i h_i}{\sum m_i} \sum m_i g = Mgh$$

$$\therefore U_A = U_B$$

② 입자계에서의 뉴턴 제 2법칙

(Newton's 2nd law for a system of particles)



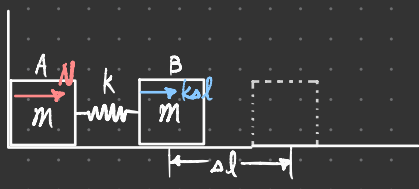
$$\vec{F}_{tot} = \sum_i \vec{F}_i = \sum_i m_i \vec{a}_i$$

$$= \sum_i m_i \ddot{\vec{r}}_i = \sum_i \frac{d^2}{dt^2} (m_i \vec{r}_i)$$

$$= \frac{d^2}{dt^2} (\sum_i m_i \vec{r}_i) = \frac{d^2}{dt^2} M \vec{r}_{cm}$$

$$= M \ddot{\vec{a}}_{cm}$$

예제) 스프링-동수철 계
(carts - spring system)



(i) 손을 떼는 직후 a_A, a_B, a_{CM}

$$a_A = 0$$

$$a_B = \frac{k\Delta l}{m}$$

$$a_{CM} = \frac{F_{ext}}{M} = \frac{N}{2m} = \frac{k\Delta l}{2m}$$

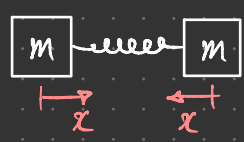
(ii) A가 벽에서 분리될 때까지 걸린 시간 t

$$t = \frac{T}{4} = \frac{1}{4} \cdot 2\pi \sqrt{\frac{m}{k}} = \frac{\pi}{2} \sqrt{\frac{m}{k}}$$

(iii) 계가 등속운동할 때 진동주기 T'

$$T' = 2\pi \sqrt{\frac{m}{2k}}$$

<C.M. Frame>



$$m\ddot{x} = -k(2x)$$

$$\therefore \omega = \sqrt{\frac{2k}{m}} \text{의 단진동}$$

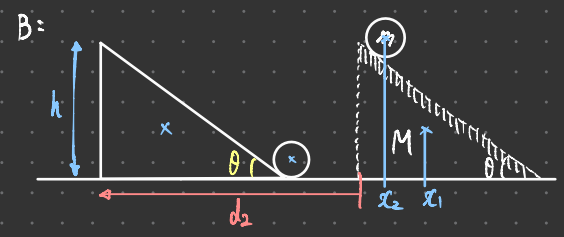
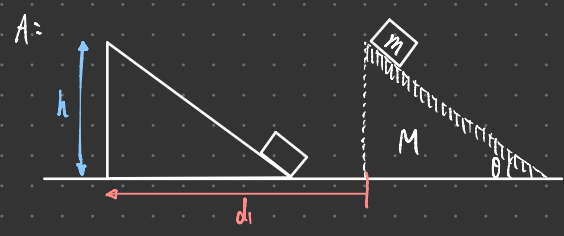
$$\therefore T' = 2\pi \sqrt{\frac{m}{2k}}$$



cf.) 동수철 절반 $2k$
C.M. Frame에서
C.M.은 정지. $\Rightarrow$ 벽 생각

$$\therefore T' = 2\pi \sqrt{\frac{m}{2k}}$$

예제) 빗면 위의 상자와 공



수평방향 외력 없으므로 $\Delta x_{CM} = 0$

$$x_{CM} = \frac{Mx_1 + mx_2}{M+m}$$

$$x'_{CM} = \frac{M(x_1 - d_2) + m(x_2 - d_2 + h \cot \theta)}{M+m}$$

$$= x_{CM} - d_2 + \frac{m}{M+m} h \cot \theta = x_{CM}$$

$$\Rightarrow d_2 = \frac{m}{M+m} h \cot \theta = d_1$$

$$\therefore d_1 = d_2$$

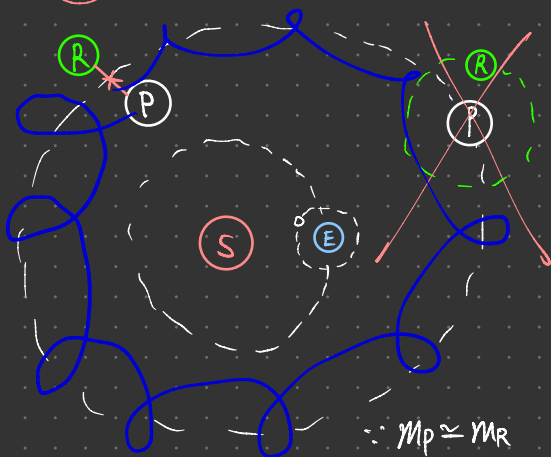
평왕성 되를 이유?

지구는 태양 중심 공전운동 $\Rightarrow \times$

지구, 태양의 질량중심이 중심임.

[행성의 조건]

1. 항성 주위를 공전
2. 자체 중력으로 구형 유지
3. 자기공전케로부터 지배적인 역활하는 천체



질량중심의 운동에 집중!!



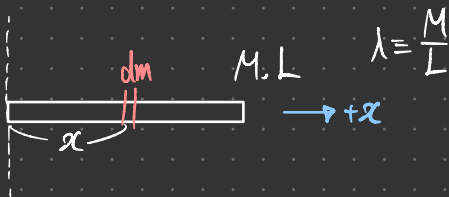
③ 연속분포된 물질 (continuous matter)의 질량중심

$$\vec{r}_{CM} = \frac{\sum m_i \vec{r}_i}{M} : \text{입자계}$$

$$\vec{r}_{CM} = \frac{\int \vec{r} dm}{M} : \text{연속입자계}$$

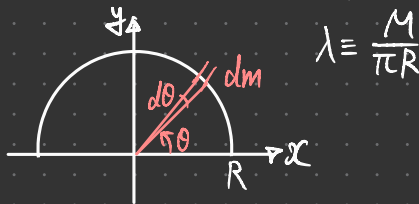
$$x_{CM} = \int x dm \quad y_{CM} = \int y dm$$

i) 균일한 막대 (rod)



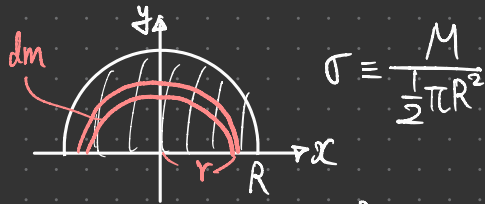
$$\begin{aligned} x_{CM} &= \frac{1}{M} \int x dm = \frac{1}{M} \int_0^L x \cdot \lambda dx \\ &= \frac{\lambda}{M} \times \frac{L^2}{2} = \left(\frac{1}{2} L \right) \end{aligned}$$

ii) 반원 고리 (half-ring)



$$\begin{aligned} y_{CM} &= \frac{1}{M} \int y dm = \frac{1}{M} \int_0^\pi R \sin \theta \cdot \lambda R d\theta \\ &= \frac{R^2}{\pi R} \times 2 = \left(\frac{2R}{\pi} \right) \end{aligned}$$

iii) 반원판 (half-disk)



$$y_{CM} = \frac{1}{M} \int y dm = \frac{1}{M} \int_0^R \frac{2r}{\pi} \cdot \sigma \cdot \pi r dr$$

$$= \frac{2\sigma}{M} \times \frac{R^3}{3} = \left(\frac{4}{3\pi} R \right)$$

by Pappus' theorem.

i) 반원 고리

$$2\pi \cdot \pi R \cdot y_{CM} = 4\pi R^2$$

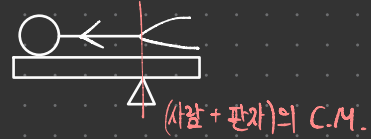
$$\therefore y_{CM} = \frac{2R}{\pi}$$

ii) 반원 판

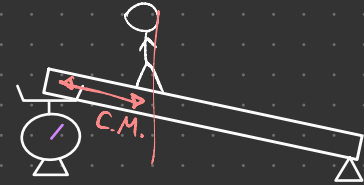
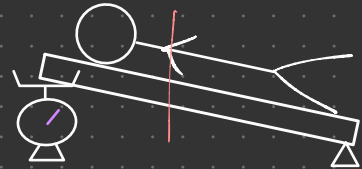
$$2\pi \cdot \frac{1}{2}\pi R^2 \cdot y_{CM} = \frac{4}{3}\pi R^3$$

$$\therefore y_{CM} = \frac{4}{3\pi} R$$

예제) 사람의 질량 중심

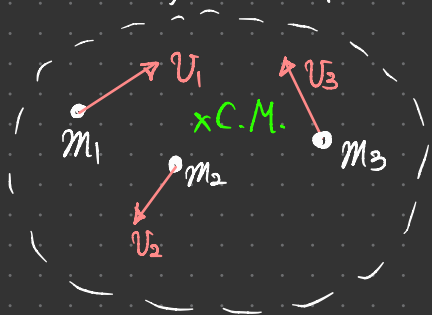


사람의 C.M. 구하는 법은??



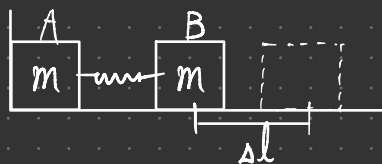
2. 선운동량

① 입자계의 운동량 (Linear momentum for a system of particle)



$$\begin{aligned}\vec{P}_{\text{tot}} &= \sum \vec{p}_i = \sum m_i \vec{v}_i \\ &= \sum m_i \frac{d\vec{r}_i}{dt} = \frac{d}{dt} (\sum m_i \vec{r}_i) \\ &= \frac{d}{dt} M \vec{r}_{\text{cm}} = M \vec{v}_{\text{cm}}\end{aligned}$$

예제) 수레 - 용수철 계 2



A가 벽에서 분리되는 순간 $\vec{v}_{\text{cm}}$

$$\begin{aligned}\vec{v}_{\text{cm}} &= \frac{\vec{P}_{\text{tot}}}{M} = \frac{\vec{P}_B}{2m} \quad (\because v_A = 0) \\ &= \frac{1}{2} \vec{v}_B\end{aligned}$$

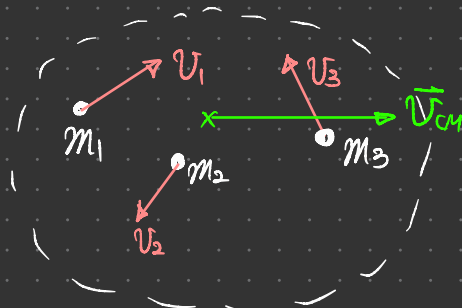
$$\Delta E = 0: \frac{1}{2} k \Delta l^2 = \frac{1}{2} m v_B^2$$

$$\Rightarrow v_B = \Delta l \sqrt{\frac{k}{m}}$$

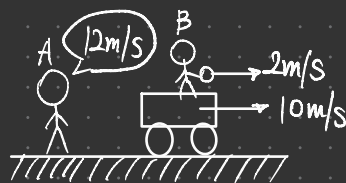
$$\therefore \vec{v}_{\text{cm}} = \frac{1}{2} \Delta l \sqrt{\frac{k}{m}}$$

② 입자계의 운동에너지

(Kinetic energy for system of particles)



$$\vec{v}_i = \vec{v}_{\text{cm}} + \vec{v}_{i,\text{cm}}$$



$$\begin{aligned}K_{\text{tot}} &= \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i (\vec{v}_i \cdot \vec{v}_i) \\ &= \sum_i \frac{1}{2} m_i (\vec{v}_{\text{cm}} + \vec{v}_{i,\text{cm}}) \cdot (\vec{v}_{\text{cm}} + \vec{v}_{i,\text{cm}}) \\ &= \sum_i \frac{1}{2} m_i v_{\text{cm}}^2 + \sum_i \frac{1}{2} m_i v_{i,\text{cm}}^2 \\ &\quad + \sum m_i (\vec{v}_{\text{cm}} \cdot \vec{v}_{i,\text{cm}})\end{aligned}$$

$$\begin{aligned}
 & \sum_i m_i (\vec{V}_{CM} - \vec{V}_{i,CM}) \\
 &= \vec{V}_{CM} \cdot \sum m_i \vec{V}_{i,CM} \\
 &= \vec{V}_{CM} \cdot \sum m_i (\vec{V}_i - \vec{V}_{CM}) \\
 &= \vec{V}_{CM} \cdot (\sum m_i \vec{V}_i - \sum m_i \vec{V}_{CM}) \\
 &= \vec{V}_{CM} \cdot (\vec{P}_{tot} - M \vec{V}_{CM}) = 0
 \end{aligned}$$

$$\begin{aligned}
 K_{tot} &= \frac{1}{2} M V_{CM}^2 + \sum_i \frac{1}{2} m_i V_{i,CM}^2 \\
 &= K_{CM} + K_{rel} \quad \left(\begin{aligned} \vec{F}_{tot} &= M \vec{a}_{CM} \\ \vec{P}_{tot} &= M \vec{V}_{CM} \end{aligned} \right)
 \end{aligned}$$

König's Theorem

예제) 수레 - 용수철 계 3

용수철이 최대압축 길이 $d = \frac{\sqrt{2}}{2} \Delta l$



벽에서 분리된 순간 용수철은 자연길이

$$K_{tot} = \frac{1}{2} k (\Delta l)^2$$

$$\begin{aligned}
 K_{CM} &= \frac{1}{2} 2m (V_{CM})^2 = \frac{1}{2} 2m \left(\frac{\Delta l}{2} \sqrt{\frac{k}{m}} \right)^2 \\
 &= \frac{1}{4} k (\Delta l)^2
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow K_{rel} &= K_{tot} - K_{CM} = \frac{1}{4} k (\Delta l)^2 \\
 &\quad \hookrightarrow \text{전부 11평으로 전환} \quad \frac{1}{2} k d^2 \\
 \therefore d &= \frac{\sqrt{2}}{2} \Delta l
 \end{aligned}$$

cf)

$$\begin{aligned}
 K_{rel} &= \sum_i \frac{1}{2} m_i V_{i,CM}^2 \\
 &= \frac{1}{2} m_A V_{A,CM}^2 + \frac{1}{2} m_B V_{B,CM}^2 \\
 &= \frac{1}{2} m (V_A - V_{CM})^2 + \frac{1}{2} m (V_B - V_{CM})^2 \\
 &= \frac{1}{2} m (0 - V_{CM})^2 + \frac{1}{2} m (2V_{CM} - V_{CM})^2 \\
 &= m V_{CM}^2 = \frac{1}{4} k (\Delta l)^2
 \end{aligned}$$



파리 방향 꺾임, 파리 정지 $\Rightarrow$ 기차도 정지해야 함



제그러려야 함.

3. 충돌 (Collision)

① 탄성 충돌 (elastic collision)

$$(m_1) \rightarrow \vec{v}_1 \quad (m_2) \rightarrow \vec{v}_2$$

$$\begin{aligned}
 \sum m_i v_i &= m_1 v_1 + m_2 v_2 \\
 \left\{ \begin{aligned} \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 &= \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2 \end{aligned} \right.
 \end{aligned}$$

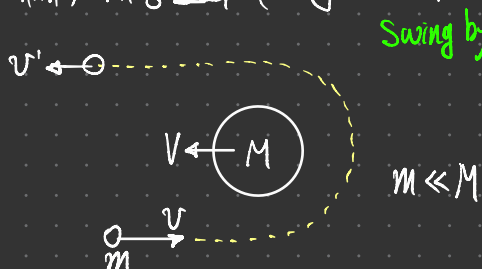
$$\Rightarrow \begin{cases} m_1 (v_1 - v_1') = m_2 (v_2' - v_2) \\ m_1 (v_1^2 - v_1'^2) = m_2 (v_2'^2 - v_2^2) \end{cases}$$

$$\Rightarrow v_1 + v_1' = v_2' + v_2$$

$$\begin{pmatrix} v_1' = \boxed{} v_1 + \boxed{} v_2 \\ v_2' = \boxed{} v_1 + \boxed{} v_2 \end{pmatrix}$$

$$\begin{cases} v_2' = v_1 + v_1' - v_2 \\ v_1' = v_2' + v_2 - v_1 \end{cases} \text{ 을 대입 후 정리}$$

예제) 새총 효과 (sling shot effect)

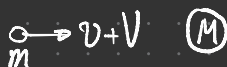


탄성충돌한다고 가정

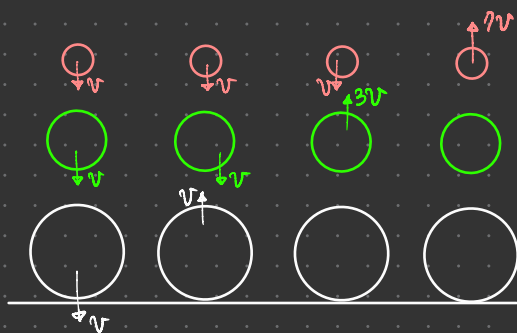
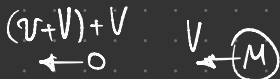
$$\begin{aligned} \text{i) } v' &= \frac{m-M}{m+M} v + \frac{2M}{m+M} (-V) \\ &\simeq -v - 2V = -(v+2V) \end{aligned}$$

ii) 공식 사용 X. 상대운동

M 입장



지구 입장



② 비탄성 충돌 (inelastic collision)

실험실계 :

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2 + Q$$

질량 중심계 :

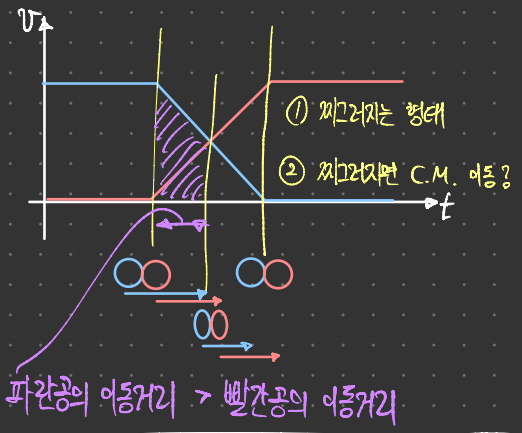
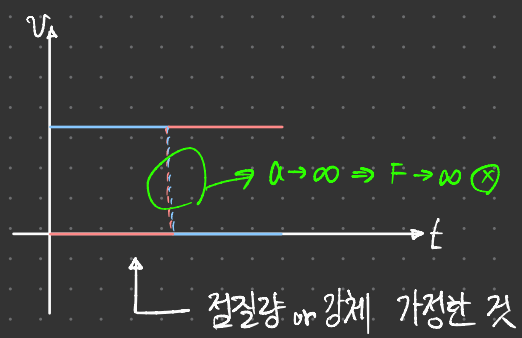
$$\frac{1}{2} m_1 v_{1,cm}^2 + \frac{1}{2} m_2 v_{2,cm}^2 = \frac{1}{2} m_1 v_{1,cm}'^2 + \frac{1}{2} m_2 v_{2,cm}'^2 + Q_{cm}$$

$$\begin{cases} Q = Q_{cm} \\ v_{i,cm} = v_i - v_{cm} \\ v_{i,cm}' = v_i' - v_{cm} \end{cases}$$

$$\begin{aligned} &\frac{1}{2} m_1 v_{cm}^2 + \frac{1}{2} m_2 v_{cm}^2 + m_1 v_i v_{cm} + m_2 v_2 v_{cm} \\ &= \frac{1}{2} m_1 v_{cm}'^2 + \frac{1}{2} m_2 v_{cm}'^2 + m_1 v_i' v_{cm} + m_2 v_2' v_{cm} \end{aligned}$$

$$\Rightarrow m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$

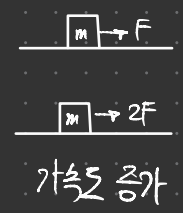
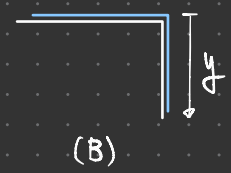
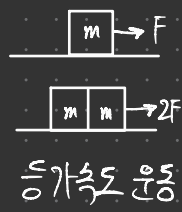
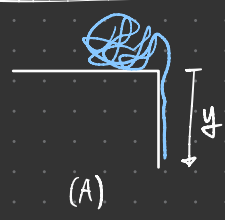
운동량 보존.



4. 질량이 변하는 계 (system of varying mass)

① 탁자 위의 줄

M, L, λ . frictionless



i) 물친 줄

(1) 운동방정식 (equation of motion)

$$F = \frac{dp}{dt}$$

$$(1/2)g = \frac{d}{dt}(1/2 v)$$

$$g = v \frac{dv}{dt} + \frac{dv}{dt} v$$

$$= v^2 + v \frac{dv}{dt}$$

(코시-오일러 미분방정식)

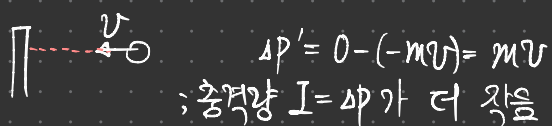
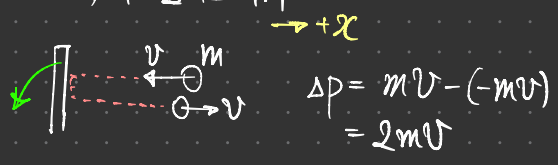
예제) 운동량과 충격량 (impulse)

i) 계산

$$I = \Delta p = F \Delta t \quad \left(\begin{array}{l} \text{계산 천으로 된} \\ \text{벽에 던지기} \end{array} \right)$$

Δt 커서 $F \downarrow$

ii) 막대 넘버트리기



$$v = Ay^n \text{ 대입}$$

$$gy = A^2 y^{2n} + y(Ay^n)(Any^{n-1})$$

$$= A^2 y^{2n} + A^2 n y^{2n}$$

$$= A^2(n+1)y^{2n}$$

$$\therefore n = \frac{1}{2}, A = \sqrt{\frac{2}{3}g}$$

$$\therefore v = \sqrt{\frac{2}{3}gy}$$

$$a = \frac{dv}{dt} = v \frac{dv}{dy}$$

$$= \sqrt{\frac{2}{3}gy} \times \frac{1}{2} \left(\frac{2}{3}gy \right)^{-\frac{1}{2}} \cdot \left(\frac{2}{3}g \right) = \frac{1}{3}g$$

(2) 차원분석 (dimensional analysis)

$$y = Bgt^2$$

$$v = \dot{y} = 2Bgt$$

$$a = \dot{v} = 2Bg$$

$$gy = v^2 + y \dot{v} \text{ 에 대입}$$

$$g(Bgt^2) = 4B^2g^2t^2 + (Bgt^2)(2Bg)$$

$$Bgt^2 = 6B^2gt^2$$

$$\therefore B = \frac{1}{6}$$

$$y = \frac{1}{6}gt^2$$

$$v = \frac{1}{3}gt = \frac{1}{3}g \sqrt{\frac{6y}{g}}$$

$$a = \frac{1}{3}g$$

$$\Rightarrow v = \sqrt{\frac{2}{3}gL} \text{ (실이 다 풀렸을 때)}$$

ii) 퍼진 줄

(1) 운동방정식

$$F = \frac{dp}{dt}$$

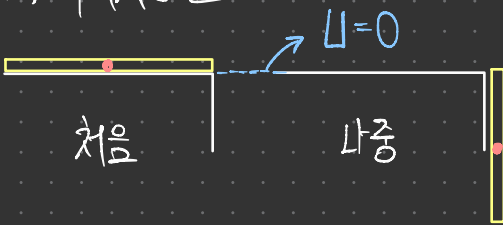
$$(\lambda y)g = \frac{d}{dt}(Mv)$$

$$\cancel{\lambda} g y = \cancel{\lambda} v \frac{dv}{dy}$$

$$\int_0^L g y dy = \int_0^v v dv$$

$$\frac{g}{2} L^2 = \frac{1}{2} v^2 \quad \therefore v = \sqrt{gL}$$

(2) 에너지 보존



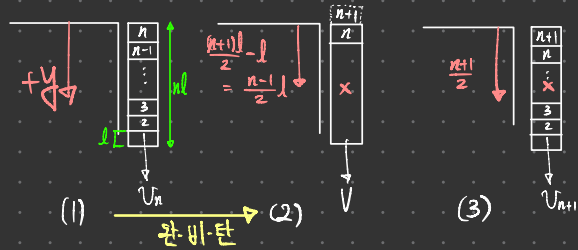
$$0 = \frac{1}{2} M v^2 - \frac{1}{2} M g L$$

$$\Rightarrow v = \sqrt{gL}$$

(A)는 왜 에너지 보존 안될까?

(A) = 완전 비탄성 충돌.

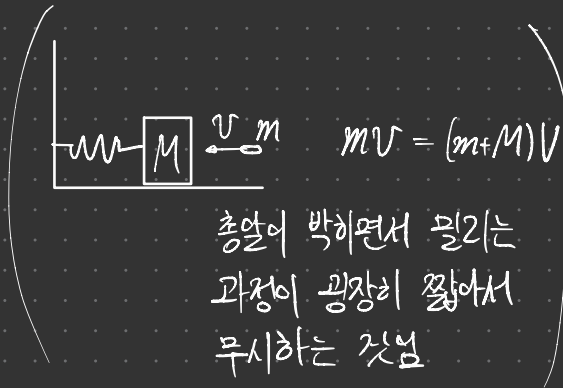
iii) 뭉친 줄에서의 에너지 손실 (energy loss)



(1 → 2) 운동량 보존 (외력 = 0 ...)

$$(\lambda n l) v_n = (\lambda (n+1) l) V$$

$$V = \frac{n}{n+1} v_n$$



(2 → 3) 에너지 보존 ($\Delta K + \Delta U = 0$)

$$\frac{1}{2} (\lambda (n+1) l) (v_{n+1}^2 - V^2) - (\lambda (n+1) l) g l = 0$$

$$\therefore v_{n+1}^2 = V^2 + 2gl$$

$$V \text{ 대입} \Rightarrow v_{n+1}^2 = \left(\frac{n}{n+1}\right)^2 v_n^2 + 2gl$$

$$= \left(1 + \frac{1}{n}\right)^{-2} v_n^2 + 2gl$$

$$\approx \left(1 - \frac{2}{n}\right) v_n^2 + 2gl \quad (n \gg 1)$$

$$\Rightarrow v_{n+1}^2 - v_n^2 = -\frac{2}{n} v_n^2 + 2gl$$

$$\Rightarrow \frac{v_{n+1}^2 - v_n^2}{l} = -\frac{2}{nl} v_n^2 + 2g$$

$$\Rightarrow \frac{dv^2}{dy} = -\frac{2}{y} v^2 + 2g$$

$$\Rightarrow \frac{1}{2} y \frac{dv^2}{dy} + v^2 = gy$$

$$cf) F = \frac{dp}{dt};$$

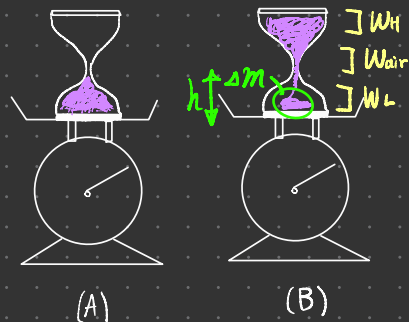
$$gy = v^2 + y \frac{dv}{dt}$$

$$= v^2 + y v \frac{dv}{dy}$$

$$\frac{dv^2}{dy} = 2v \frac{dv}{dy} \text{ 이므로 } v \frac{dy}{dy} = \frac{1}{2} \frac{dv^2}{dy}$$

$\therefore$ 동일한 미분방정식!

② 모래시계 (sandglass)



$$W_A = W_H + W_L + W_{air}$$

$$W_B = W_H + W_L + F$$

모래알에 의한 충격력

i) 가정

(1) 모래알은 바닥에 닿으면 정지

(2) 모래알은 일정한 비율로 낙하

ii) Δt 동안 바닥에 작용하는 충격력 F

(1) 모래가 바닥에 닿을 때 속력

$$v = \sqrt{2gh}$$

(2) 충격력

$$F = \frac{\Delta p}{\Delta t} = \frac{\Delta m}{\Delta t} \sqrt{2gh}$$

iii) 공기 중에 있는 모래 무게 W_{air}

(1) 모래의 질량

$$\Delta m : \Delta t = \Delta m_{air} : \sqrt{\frac{2h}{g}}$$

$$\Delta M_{air} = \frac{\Delta m}{\Delta t} \sqrt{\frac{2h}{g}}$$

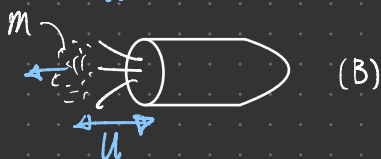
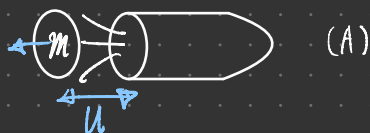
(2) 물레의 무게

$$W_{air} = \frac{\Delta m}{\Delta t} \sqrt{2gh}$$

$\therefore F = W_{air}$ 이므로 (A) 와 (B)의

눈금 동일!

③ 로켓 (rocket)



i) 한계번에 분사 (A)



$$Mv = m(v_A - u) + (M-m)v_A$$

$$v_A = \frac{Mv + mu}{M} = v + \frac{m}{M}u$$

ii) 연속으로 분사

(1) 두 번에 나뉘서 분사

1) 첫번째



$$Mv = \frac{m}{2}(v' - u) + (M - \frac{m}{2})v'$$

$$v' = v + \frac{m}{2M}u$$

2) 두 번째



$$(M - \frac{m}{2})v' = \frac{m}{2}(v'' - u) + (M - m)v''$$

$$v'' = v' + \frac{m}{2M - m}u$$

$$= v + \left(\frac{1}{2M} + \frac{1}{2M - m} \right) mu$$

(2) 세 번에 나뉘서 분사

$$v''' = v + mu \left(\frac{1}{3M} + \frac{1}{3M - m} + \frac{1}{3M - 2m} \right)$$

(3) n 번 나뉘어서 분사

$$V_n = V + mu \sum_{k=1}^n \frac{1}{nM - (k-1)m}$$

(4) 연속으로 분사

$$\begin{aligned} \lim_{n \rightarrow \infty} V_n &= V + mu \int_0^1 \frac{dx}{M - mx} \\ &= V - u \ln \left| \frac{M-m}{M} \right| \\ &= V + u \ln \left(\frac{M}{M-m} \right) \end{aligned}$$

iii) 속력 비교

$$V_A = V + \frac{m}{M} u$$

$$V_B = V + u \ln \left(\frac{M}{M-m} \right)$$

if) $-1 < x \leq 1$:

$$\ln(1+x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \dots$$

$$\ln \left(\frac{M}{M-m} \right) = -\ln \left(1 - \frac{m}{M} \right)$$

$$= - \left[-\frac{m}{M} - \frac{1}{2} \frac{m^2}{M^2} - \dots \right]$$

$$= \frac{m}{M} + \frac{m^2}{2M^2} + \dots > \frac{m}{M}$$

$$\therefore V_A < V_B$$

정성적인 설명:

연료의 운동량 (A) < (B)

$$mV < m(V_1 + V_2 + \dots)$$